

Q-1. (a) If  $f$  is a character of a finite abelian group  $G$  with identity element  $e$ , then [4] show that  $f(e)=1$ .

(b) Let  $G$  be a finite abelian group of order  $n$  and  $\hat{G}$  be the set of its  $n$  distinct characters. If multiplication of characters  $f_i$  and  $f_j$  is defined by the relation  $(f_i f_j)(a) = f_i(a) f_j(a)$ , for every  $a \in G$ , then show that  $\hat{G}$  forms [10] an abelian group with this operation.

OR

Q-1. (a) Find all characters for group of order 5. [7]

(b) Let  $G = \{a_1 = e, a_2, \dots, a_n\}$  be a finite abelian group of order  $n$  and [7]

$\hat{G} = \{f_1, f_2, \dots, f_n\}$  be the group of its  $n$  distinct characters of  $G$ . Prove that

$$\sum_{r=1}^n f_r(a_i) \bar{f}(a_j) = n \text{ if } i = j.$$

Q-2. (a) Find all Dirichlet characters modulo 8. [7]

(b) Let  $\chi$  be real-valued character modulo  $k$ . Prove that  $\sum_{d|n} \chi(d) \geq 0$  for every [7] positive integer  $n$ .

OR

Q-2. (a) Let  $\chi$  be nonprincipal Dirichlet character modulo  $k$ . Show that [4]

$$\sum_{n \leq x} \frac{\chi(n) \log n}{n} = \sum_{n=1}^{\infty} \frac{\chi(n) \log n}{n} + O(x^{-1} \log x) \text{ for all } x > e.$$

(b) Prove that  $L(1, \chi) \neq 0$  for any nonprincipal Dirichlet character  $\chi$  modulo  $k$ . [10]

Q-3. (a) For nonprincipal Dirichlet character  $\chi$  mod  $k$ , show that [7]

$$L(1, \chi) \sum_{n \leq x} \frac{\mu(n) \chi(n)}{n} = O(1) \text{ for all } x > 1.$$

(b) For nonprincipal Dirichlet character  $\chi$  mod  $k$ , show that [7]

$$\sum_{\substack{p \leq x \\ p, \text{ prime}}} \frac{\chi(p) \log p}{p} = -L'(1, \chi) \sum_{n \leq x} \frac{\mu(n) \chi(n)}{n} + O(1) \text{ for all } x > 1.$$

OR

- Q-3. (a) Let  $\chi$  be nonprincipal Dirichlet character mod  $k$ . If  $L(1, \chi) = 0$  then, show [7] that  $L'(1, \chi) \sum_{n \leq x} \frac{\mu(n)\chi(n)}{n} = \log x + O(1)$  for all  $x > 1$ .
- (b) If  $\gcd(h, k) = 1$ , then show that the arithmetical progression  $\{nk + h\}_{n=1}^{\infty}$  [7] contains infinitely many primes.

- Q-4. State and prove Lagrange's interpolation theorem. [14]

OR

- Q-4. (a) Let  $r_k(n) = \sum_{d|(n,k)} \frac{k^2}{d}$ . If  $\gcd(a, k) = 1$ , then show that  $r_{mk}(a) = r_m(a)k^2$ . [7]
- (b) Let  $\chi$  be any Dirichlet character modulo  $k$ . If  $\gcd(n, k) = 1$ , then show that [7]  $G(n, \chi) = \overline{\chi}(n)G(1, \chi)$ .

- Q-5. (a) If  $G(n, \chi)$  is separable for every  $n$ , then show that  $|G(1, \chi)|^2 = k$ . [7]
- (b) Let  $\chi$  be a Dirichlet character mod  $k$ . Prove:  $\chi$  is primitive mod  $k$  if and [7] only if the Gauss sum  $G(n, \chi)$  is separable for every  $n$ .

OR

- Q-5. If  $\chi$  is any primitive character mod  $k$ , then prove that [14]

$$\left| \sum_{m \leq x} \chi(m) \right| < \sqrt{k} \log k \text{ for all } x \geq 1.$$