CODE : 8402

S. Y. B. Sc Mathematics Examination March/April-2016
Linear Algebra
Paper No: M-202

Total Marks: 100

Time : 3 hours
Paper code:

Instructions:

(1) All questions are compulsory.
(2) Each question carry equal marks.

Que-1. (a) Let $V = \{(x, y) : y \in \mathbb{R}; y > 0\}$. For $(a, b), (c, d) \in V$ and $\alpha \in \mathbb{R}$, $(a, b) + (c, d) = (a + c, bd)$ and $\alpha(a, b) = (\alpha a, b^{\alpha})$. Show that $(V, +, \cdot)$ a vector space over $\mathbb{R}$. [7]

(b) Extend the set $S = \{(1, 1, 1), (2, 0, 0)\}$ to the basis of $\mathbb{R}^3$. [7]

(c) Let W_1 and W_2 be two subspaces of a vector space V. Then show that $W_1 \cap W_2$ is a subspace of V. [6]

or

Que-1. (a) Define *basis* in vector space. Prove that $B = \{1, x, x^2 + x, x^3 + 3x^2 + 2x\}$ is a basis of $\mathcal{P}_3(\mathbb{R})$, where $\mathcal{P}_3(\mathbb{R})$ denotes the set of all polynomials of degree less than or equal to three defined on $\mathbb{R}$. [7]

(b) Show that $W = \{(a, b, c) \in \mathbb{R}^3 : 2a - 3b + 4c = 0\}$ is a subspace of $\mathbb{R}^3$. Find its dimension. [7]

(c) Show that any linearly independent subset of a vector space V cannot contain more elements than basis. [6]

Que-2. (a) State and prove the rank-nullity theorem. [7]

(b) Prove or disprove: There exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(1, 0, 0) = (1, 1, 0)$, $T(0, 1, 0) = (0, 1, 1)$ and $T(0, 0, 1) = (1, 0, 1)$. [7]

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T(x, y) = (2x + 3y, 4x - 5y)$, $(x, y) \in \mathbb{R}^2$. Find $T(1, 2)$ and express it as a linear combination of $\{(1, 2), (2, 5)\}$. [6]

or

Que-2. (a) Let V and W be two vector spaces and $T : V \to W$ be a linear transformation. Show that $N(T)$ is a subspace of V and $R(T)$ is a subspace of W. [7]

(b) Show that $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (2x + y, 3x + 2y)$, $(x, y) \in \mathbb{R}^2$ is a linear transformation. Is T invertible? If yes, then find its inverse. [7]

(c) Let V and W be two vector spaces and $T : V \to W$ be a linear transformation. Show that if T is one one, then it maps linearly independent set to linearly independent set. [6]

Que-3. (a) Find the rank and nullity of the linear transformation whose matrix representation relative to standard basis is given by $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 3 & 7 & 0 \end{pmatrix}$. [7]

(b) Find the matrix representation of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (2x + 3y, 4x - 5y)$, $(x, y) \in \mathbb{R}^2$, relative to the basis $\{(1, 2), (2, 5)\}$. [7]

(c) Find the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose matrix representation relative to the standard basis is given by the matrix $\begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 9 \\ 3 & 2 & -1 \end{pmatrix}$. [6]

or

Que-3. (a) With all details show that composition two linear transformations is also a linear transformation. [7]

(b) Let U and V be vector spaces and $A = \{u_1, u_2, \ldots, u_n\}$ be a basis of U. Let v_i, $1 \leq i \leq n$ be any set of (not necessarily distinct) vectors in V. Then prove that there is a unique linear map $T : U \to V$ such that $T(u_i) = v_i$. [7]

(c) Verify the rank-nullity theorem for the matrix $A = \begin{pmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{pmatrix}$ [6]

Que-4. (a) State and prove Cauchy-Schwarz inequality. [7]

(b) Show that any orthogonal set of vectors in an inner product space is linearly independent. [7]

(c) Let $\mathcal{P}(\mathbb{R})$ denotes the inner product space of all polynomials with real coefficients with inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$, $f(t), g(t) \in \mathcal{P}(\mathbb{R})$. Let $p(t) = t + 2$ and $q(t) = t^2 - 2t - 3 \in \mathcal{P}(\mathbb{R})$. Find $\langle p, q \rangle$, $\|p\|$, $\|q\|$. [6]

or

Que-4. (a) Let W be a subspace of an inner product space V. Then show that $W^{\perp}$ is also a subspace of V and $V = W \oplus W^{\perp}$. [7]

(b) Apply Gram-Schmidt orthogonalization process to find an orthonormal basis of $\mathbb{R}^3$ from the basis $\{(2, -1, -1), (0, 1, 1), (1, 1, 0)\}$. [7]

(c) State and prove triangle inequality in an inner product space. [6]

Que-5. (a) Solve the system $x + y + z = 5$, $x - 2y - 3z = -1$, $2x + y - z = 3$ using Cramer's rule. [7]

(b) For $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 6 \\ 2 & 6 & 7 \end{pmatrix}$, find a non-singular matrix P such that P^tAP is a diagonal matrix. [7]

(c) Give an example of an orthogonal linear transformation on $\mathbb{R}^3$. Prove your claim. [6]

or

Que-5. (a) Use diagonalization method to obtain the direction of the principal axes of the conic $17x^2 + 312xy + 108y^2 = 900$.

(b) Give an example of 3×3 orthogonal matrix and find its rank.

(c) For $k \in \mathbb{R}$ and an $n \times n$ matrix A, show that $|kA| = k^n|A|$, where $|A|$ denotes the determinant of A.